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Two Classes of Perfect Graphs

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We present two classes of perfect graphs. The first class is defined through a construction used by Gallai in his investigation of comparability graphs; the second class contains those Berge graphs defined through a forbidden graph on five vertices.

1. INTRODUCTION

Claude Berge proposed to call a graph *perfect* if, for each of its induced subgraphs F, the chromatic number of F equals the largest number of pairwise adjacent vertices in F. A *clique* in a graph is any set of pairwise adjacent vertices; a *stable set* is any set of pairwise nonadjacent vertices. An *odd hole* is a chordless cycle whose number of vertices is odd and at least five. Berge conjectured that a graph is perfect if and only if none of its induced subgraphs is an odd hole or the complement of an odd hole. This is known as the *Strong Perfect Graph Conjecture*. The only if part is trivial; the if part is still open. At the same time Berge made a weaker conjecture: a graph G is perfect if and only if its complement is perfect. This was proved by Lovász [11] and it is known as the Perfect Graph Theorem. Vašek Chvátal proposed to call a graph *Berge* if none of its induced subgraphs is an odd hole or the complement of an odd hole. In this terminology, the Strong Perfect Graph Conjecture states that a graph is perfect if and only if it is Berge. For more information on perfect graphs, including definitions of triangulated graphs, comparability graphs, parity graphs, etc., see Golumbic [8] or Berge and Chvátal [2].

One way to make progress in attacking the Strong Perfect Graph Conjecture is to prove that all graphs in some special class of Berge graphs are perfect. One special class of Berge graphs emerged around 1975–1976 from discussions among Claude Berge, Pierre Duchet, Michel Las Vergnas, and Henry Meyniel: these are graphs in which every cycle whose length is odd

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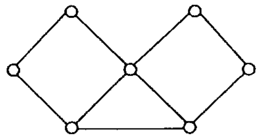


FIGURE 1

and at least five has at least one chord joining two vertices whose distance on the cycle is two (such chords are called *short* or *triangular*). Since Berge, Duchet, Las Vergnas, and Meyniel were all working at 54 Boulevard Raspail in Paris, Chvátal proposed to call these graphs *Raspail graphs*. To see that every Raspail graph is Berge, consider an odd hole $v_1 v_2 \cdots v_{2k+1} v_1$ (with $k \geqslant 2$). Trivially, the hole is not Raspail; its complement is not Raspail either, since it contains the odd cycle $v_1 v_{k+2} v_2 v_{k+3} \cdots v_k v_{2k+1} v_{k+1} v_1$ with no triangular chord. Hence all Raspail graphs are Berge; an example of a Berge graph which is not Raspail is shown in Fig. 1. The conjecture that all Raspail graphs are perfect (which is a corollary of the Strong Perfect Graph Conjecture) remains open.

Chvátal suggested restricting the class of Raspail graphs even further as follows. First, given any graph G, define a graph $\mathrm{Gal}(G)$ by letting the vertices of $\mathrm{Gal}(G)$ be the edges of G, and making two vertices of $\mathrm{Gal}(G)$ adjacent if and only if the corresponding two edges of G share an endpoint and their other two endpoints are nonadjacent in G (so that the two edges form a chordless path with three vertices in G). This construction was used by Gallai [7] in his investigation of comparability graphs; hence our notation. We call a graph G *Gallai-perfect* if, and only if, $\mathrm{Gal}(G)$ contains no odd hole. Obviously, every Gallai-perfect graph is Raspail; an example of a Raspail graph which is not Gallai-perfect is shown in Fig. 2.

THEOREM 1. *Every Gallai-perfect graph is perfect.*

The class of Gallai-perfect graphs includes all bipartite graphs (if G is bipartite then $\mathrm{Gal}(G)$ is the line graph of G), all complements of bipartite graphs, and all line-graphs of bipartite graphs (if G is the complement of a bipartite graph or the line-graph of a bipartite graph then $\mathrm{Gal}(G)$ is bipartite).

Note that the graph in Fig. 2, which is not Gallai-perfect, is triangulated; its complement is a triangulated graph, a parity graph, and the line-graph

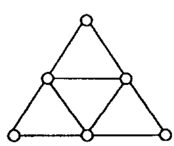


FIGURE 2

FIGURE 3

of a bipartite graph. Two additional examples of graphs that are not Gallai-perfect are shown in Fig. 3. The first is a comparability graph and the complement of a comparability graph; the second is a parity graph.

A popular way of creating special classes of Berge graphs consists of choosing a graph F and then considering F-free Berge graphs (that is, Berge graphs with no induced subgraph isomorphic to F). For instance (see Fig. 4), F-free Berge graphs are known to be perfect when F is the *claw* (Parthasarathy and Ravindra [14]), the *tetrahedron* (Tucker [18]), the *diamond* (Parthasarathy and Ravindra [15], Tucker [18]), or the *bull* (Chvátal and Sbihi [6]).

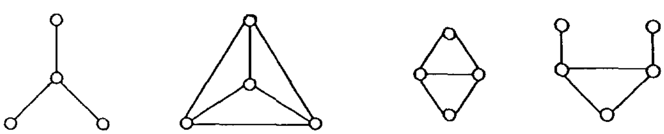
We shall prove a result of this kind, where F is the graph shown in Fig. 5; we call this graph the *dart*.

THEOREM 2. *Every dart-free Berge graph is perfect.*

Trivially, the class of dart-free Berge graphs includes all claw-free Berge graphs (which, in turn, include all line-graphs of bipartite graphs and all complements of bipartite graphs) as well as all diamond-free Berge graphs (which, in turn, include all line-graphs of bipartite graphs as well as bipartite graphs).

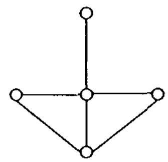
Incidentally, for each of the five graphs F shown in Fig. 4 and 5 there are F-free Berge graphs that are not Gallai-perfect: see Fig. 1, 2, or 6. (Note also that each of these five graphs F is Gallai-perfect.) However, diamond-free Raspail graphs are Gallai-perfect; we prove this in Section 4.

A *star-cutset* in a graph G is a nonempty set S of vertices such that some vertex in S is adjacent to all the remaining vertices in S and such that $G - S$ is disconnected.



claw tetrahedron diamond bull

FIGURE 4



dart

FIGURE 5

STAR-CUTSET LEMMA (Chvátal [5]). *No minimal imperfect graph contains a star-cutset.*

An *even pair* in a graph G is a pair x, y of vertices of G such that all chordless paths joining x and y in G have even numbers of edges.

EVEN PAIR LEMMA (Meyniel [12]). *No minimal imperfect graph contains an even pair.*

We shall derive Theorem 1 from Theorem 2 and the following result.

THEOREM 3. *If a Gallai-perfect graph G contains an induced dart, then G contains a star-cutset or an even pair.*

To derive Theorem 1 from Theorem 2 and Theorem 3, consider an arbitrary Gallai-perfect graph G. Since all induced subgraphs of G are Gallai-perfect, we only need show that G is not minimal imperfect. If G is dart-free, then the desired conclusion follows from Theorem 2; else it follows from Theorem 3, the Star-Cutset Lemma, and the Even Pair Lemma.

Hence we only need prove Theorem 2 (which will be done in Section 2) and Theorem 3 (which will be done in Section 3).

We assume familiarity with basic notions of graph theory (see, for instance, Berge [1] or Bondy and Murty [4]). As usual, we let $\alpha(G)$ denote the largest size of a stable set in G, we let $\omega(G)$ denote the largest size of a clique in G, we let $\bar{G}$ denote the complement of G, and we let $N(u)$ denote the set of vertices in G that are adjacent to vertex u of G. Finally, we let C_n denote the chordless cycle of n vertices.

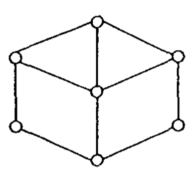


FIGURE 6

2. Dart-free Berge Graphs are Perfect

A *paw* is the graph with vertices *a*, *b*, *c*, *d* and edges *ab*, *ac*, *ad*, *cd*. This graph is shown in Fig. 7.

Lemma 1 (Olariu [13]). *Let H be a connected graph. If H is neither complete multipartite nor triangle-free, then H contains an induced paw.* ▌

We shall also use the following fact.

Lemma 2. *Every minimal imperfect Berge graph contains a vertex w such that w is the center of a claw and N(w) does not consist of vertex-disjoint cliques.*

This lemma is a special case of a theorem of Hsu ([10], Theorem 4). However, Hsu's proof is not entirely clear: in particular, in the proof of Lemma 5, the claim that $\omega(H)=3$ seems incorrect (Professor Hsu informed me that the error can be corrected along the lines used at the top of p. 195 in [9]). For this reason, we shall provide an alternative proof of Lemma 2. (Most of our proof follows Hsu's argument; we shall point out the differences later.)

Proof of Lemma 2. Let G be a Berge minimal imperfect graph; let E denote the set of edges of G; let A be the set of all vertices u in G such that u is the center of a claw; let B be the set of all vertices u in G such that $N(u)$ does not consist of vertex-disjoint cliques. We only need prove that $A \cap B \neq \emptyset$.

Since claw-free Berge graphs are perfect [14], we have $A \neq \emptyset$; since diamond-free Berge graphs are perfect [15, 18], we have $B \neq \emptyset$. We may assume that at least one vertex of G lies outside A (else $A \cap B = B \neq \emptyset$ and we are done); Since G is minimal imperfect, it is connected; it follows that some vertex u outside A has a neighbor v in A.

Write $\omega = \omega(G)$. Since K_4-free Berge graphs are perfect [17], we have $\omega \geqslant 4$. Since $G-u$ is perfect, it can be ω-colored. Let g be an ω-coloring of $G-u$. Let H be the subgraph of G induced by $N(u)$. Since H is perfect, it can be $\omega(H)$-colored. Note that $\omega(H)=\omega-1$. Let h be an $\omega(H)$-coloring of H. Since $\alpha(H)=2$ by assumption, at most two vertices in H can have one color in any coloring. Let M_g be the matching in $\bar{H}$ consisting of

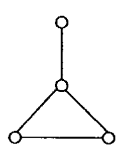


Figure 7

all edges st such that $g(s)=g(t)$ and let M_h be the matching in $\bar{H}$ consisting of all edges st such that $h(s)=h(t)$. Since g uses ω colors in H, we have $|M_g|=|N(u)|-\omega$; since h uses $\omega-1$ colors in H, we have $|M_h|=|N(u)|-\omega+1$. Hence M_g is not a maximum matching. It is well-known and easy to prove that a matching is not maximum if and only if it admits an augmenting path. In case of M_g, this implies that $\bar{H}$ contains a path $x_1y_2x_2y_3\cdots y_{k-1}x_{k-1}y_k$ such that $g(x_i)=i$ whenever $1\leqslant i<k-1$, $g(y_j)=j$ whenever $1<j\leqslant k$, and such that x_1 is the only vertex g-colored 1 in H, and y_k is the only vertex g-colored k in H. Let P denote such a path with k as small as possible. We propose to show that

$$P \text{ consists of all neighbors of } u. \tag{1}$$

For this purpose, consider the subgraph F of G induced by vertex u and all vertices g-colored 1, 2, ..., k. Let C be any clique in F. If $u\notin C$, then $|C|\leqslant k$ (since C is g-colored by k colors). If $u\in C$, then $C-u\subseteq P$, which implies that $|C-u|\leqslant k-1$ (since P is covered by $k-1$ stable sets, namely, $x_1y_2, x_2y_3, ..., x_{k-2}y_{k-1}, x_{k-1}y_k$), and so $|C|\leqslant k$. Hence $\omega(F)\leqslant k$. We know that $G-F$ is g-colored by $\omega-k$ colors. Since G cannot be colored by ω colors, we conclude that F cannot be colored by k colors. Since $\omega(F)\leqslant k$, this implies that F is imperfect, and so $F=G$. Hence $\omega=\omega(F)\leqslant k\leqslant\omega$, and so $\omega=k$. Thus (1) holds true.

Since $\alpha(H)=2$ and H is perfect, $\bar{H}$ is bipartite; hence

$$x_1, x_2, ..., x_{k-1} \text{ and } y_2, y_3, ..., y_k \text{ induce cliques in } G; \tag{2}$$

minimality of k guarantees that

$$x_i y_j \in E \qquad \text{whenever } i\leqslant j-2. \tag{3}$$

We propose to show that

$$y_2x_3\in E \qquad \text{and} \qquad x_{k-1}y_{k-2}\in E. \tag{4}$$

For this pupose, we only need verify that $y_2x_3\in E$ (by symmetry, $x_{k-1}y_{k-2}\in E$ will follow).

Since the subgraph of G induced by all vertices g-colored 2 or 3 is connected [3], it contains a chordless path Q between x_2 and y_3. Since $Q\cup\{z\}$ cannot induce an odd hole in G, Q must include y_2 or x_3. If $y_2\in Q$ then enumerate the vertices of Q as $z_1, z_2, ..., z_t$ in such a way that $z_1=y_3$, $z_2=y_2$, $z_t=x_2$, and set $w=x_1$, $z^*=x_3$; if $y_2\notin Q$, then enumerate the vertices of Q as $z_1, z_2, ..., z_t$ in such a way that $z_1=x_2$, $z_2=x_3$, $z_t=y_3$, and set $w=y_4$, $z^*=y_2$. Note that t is even. In either case, $wz_1\in E$, $wz_t\in E$, and $wz_2\notin E$; note also that $w\in B$ (since $N(x_1)$ includes x_2, x_3, y_4 and $N(y_4)$ includes x_2, y_3, x_1). Hence we may assume that $w\notin A$ (else we are done) and so $wz_i \notin E$ whenever $3 \leqslant i \leqslant t-2$ (else $wz_1z_tz_i$ is a claw); since $Q \cup \{w\}$ cannot induce an odd hole in G and since $wz_2 \notin E$, we now have $wz_{t-1} \in E$. Since $wz^* \in E$ but $wz_1z_{t-1}z^*$ is not a claw, we must have $z^* = z_{t-1}$. Hence $\{z_2, z_{t-1}\} = \{y_2, x_3\}$. Since $uz_2z_3 \cdots z_{t-1}u$ is not an odd hole, we must have $y_2x_3 \in E$.

Now we know that $P \subseteq B$:

if $i \leqslant k-2$, then $N(x_i)$ includes u, x_{k-1}, y_k (see (2), (3));

if $j \geqslant 3$, then $N(y_j)$ includes u, x_1, y_2 (see (2), (3));

$N(x_{k-1})$ includes u, y_{k-2}, x_{k-2} (see (2), (4));

$N(y_2)$ includes u, y_3, x_3 (see (2), (4)).

In particular, $v \in A \cap B$. ▮

The major difference between our proof and Hsu's argument is that we establish that $\omega = k$ (and so we have (2)), whereas Hsu shows only $k \geqslant 4$, and then relies on his Lemma 5; a minor difference is that we avoid the use of König's theorem in showing that M_g is not a maximum matching.

Proof of Theorem 2. We only need show that every minimal imperfect Berge graph contains an induced dart. For this purpose, consider an arbitrary minimal imperfect Berge graph G. By Lemma 2, G contains a vertex w such that w is the center of a claw and $N(w)$ does not consist of vertex-disjoint cliques. Let H denote the complement of the subgraph of G induced by $N(w)$. By the Perfect Graph Theorem, $\bar{G}$ is a minimal imperfect graph; now the Star-Cutset Lemma guarantees that H is connected. By Lemma 1, H contains an induced paw; hence G contains an induced dart. ▮

3. Gallai-Perfect Graphs Are Perfect

Proof of Theorem 3. It will be convenient to prove the statement in the following form:

> Let G be a graph with no star-cutset and no even pair; if G contains an induced dart, then Gal(G) contains an odd hole. (5)

Throughout the proof, G denotes a graph with no star-cutset and no even pair, and E denotes the set of edges of G. The proof consists of six claims, and the proof of each claim is based on the truth of preceding claims. The last of the six claims is precisely (5).

The proofs of the claims amounts to a tedious case-analysis; its more detailed version appears in [16]. To save space, we shall often write "G has an F on a, b, c, ...," when we mean that the subgraph of G induced by $\{a, b, c, ...\}$ is isomorphic to F.

We shall often rely on the observation that as long as G is not complete, for every two vertices x, y of G, there is a vertex t such that $tx \in E$, $ty \notin E$. (If there were no such t then either G would have a star-cutset consisting of y and some of its neighbors, or x and y would be adjacent to all the vetices distinct from x, y, and constitute an even pair.)

CLAIM 1. *If G contains an induced F_1 (see Fig. 8), then Gal(G) contains an odd hole.*

Proof. There are vertices t and t' with $te \in E$, $tb \notin E$, $t'f \in E$, and $t'b \notin E$. If $tf \notin E$ and $t'e \notin E$, then Gal(G) has a C_7 on fb, bc, db, be, et, ef, ft'; hence symmetry allows us to assume that $tf \in E$. Now if $tc \notin E$ and $td \notin E$, then Gal(G) has a C_9 on tf, fb, bc, db, be, et, ce, ef, fd. On the other hand, if $tc \in E$ and $td \in E$, then Gal(G) has a C_7 on ft, tc, dt, te, eb, ba, bf; hence symmetry allows us to assume that $td \in E$ and $tc \notin E$. To summarize, we now have

$$tb \notin E, tc \notin E, td \in E, te \in E, tf \in E.$$

There exists a chordless path $c_1 c_2 \cdots c_k$ with $c_1 = c$, $c_k = a$, and

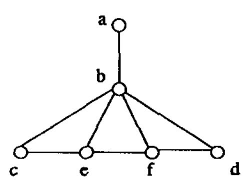
$$c_i \neq b, \quad c_i b \notin E \qquad \text{whenever } 1 < i < k.$$

We shall distinguish among three cases. In Case 1, $c_2 e \in E$, $c_2 f \in E$; in Case 2, $c_2 e \in E$, $c_2 f \notin E$; in Case 3, $c_2 e \notin E$.

Case 1. Gal(G) has a C_7 on fc_2, $c_2 c$, dc_2, $c_2 e$, eb, ba, bf or a C_9 on $c_2 f$, fd, fe, ec, te, eb, bd, bc, cc_2.

Case 2. There is a vertex s with $s \neq e$, $sc \in E$, and $se \notin E$. Now Gal(G) has a C_5 on bc, cc_2, $c_2 t$, tf, fb or a C_7 on $c_3 c_2$, $c_2 e$, et, td, db, bc, cc_2 or a C_7 on be, ec_2, fe, ec, cs, cb, ba or a C_7 on ec, cs, $c_2 c$, cb, bd, dt, te or a C_7 on cc_2, $c_2 c_3$, sc_2, $c_2 e$, eb, ba, bc or a C_7 on $c_2 s$, sb, $c_3 s$, sc, ce, ef, ec_2.

Case 3. Since Gal(G) has a C_{k+1} on bc, cc_2, ..., $c_{k-1}a$, ab when $k \geqslant 3$, we may assume that k *is odd*. Next, we may assume that $t = c_i$ for some i



F_1

FIGURE 8

with $1 < i < k$ (else Gal(G) has a C_{k+4} on ec, $cc_2, ..., c_{k-2}c_{k-1}$, $c_{k-1}a$, ab, bd, dt, te). Let j be the smallest subscript with $j > 1$ and $ec_j \in E$; note that $3 \leqslant j \leqslant i$. Since Gal$(G)$ has a C_{j+1} on cc_2, $c_2c_3, ..., c_{j-1}c_j$, c_je, ec, we may assume that j *is odd*. Next, we may assume that $c_{j-1}f \notin E$ (else Gal(G) has a C_7 on fe, ec, te, eb, ba, bf, fc_{j-1}), and then that $c_jf \notin E$ (else Gal(G) has a C_7 on fb, bc, db, be, ec_j, c_jc_{j-1}, c_jf), and then that $j < i$ (else Gal(G) has a C_{j+2} on bc, $cc_2, ..., c_{j-2}c_{j-1}$, $c_{j-1}t$, tf, fb); in fact we may assume that $j = i - 1$ (else Gal(G) has a C_{j+4} on bc, $cc_2, ..., c_{j-1}c_j$, c_je, et, td, db). Now Gal(G) has a C_7 on ft, tc_j, dt, te, eb, ba, bf or a C_7 on fd, dc_j, bd, dt, te, ec, ef. ∎

CLAIM 2. *If G contains an induce F_2 (see Fig. 9), then* Gal(G) *contains an odd hole.*

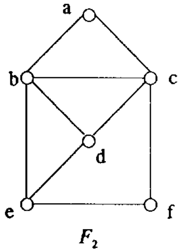
Proof. There is a vertex t with $t \neq b$, $tb \notin E$, and $td \in E$. We may assume that $tc \in E$ and $ta \in E$: otherwise Gal(G) has a C_7 on db, ba, eb, bc, cf, cd, dt or a C_7 on dc, ca, tc, cb, be, ef, ed. We shall distinguish among three cases. In Case 1, $te \in E$, $tf \in E$; in Case 2, $te \in E$, $tf \notin E$; in Case 3, $te \notin E$.

Case 1. There is a vertex s with $s \neq c$, $sa \in E$, and $sc \notin E$. We may assume that $sb \in E$, $sd \in E$, $st \in E$, $sf \in E$, and $se \notin E$: otherwise Gal(G) has a C_7 on sa, ab, be, ef, ed, dc, ca or a C_7 on cb, bs, db, ba, at, te, tc or a C_7 on ca, as, ta, ab, be, bc, cf or a C_7 on ct, ts, ft, ta, ab, be, bc or a C_9 on sa, ac, cd, de, ef, eb, bc, bs, sf.

There is a vertex u with $u \neq d$, $uf \in E$, and $ud \notin E$. We shall distinguish between two subcases. In Subcase 1.1, $ut \in E$; in Subcase 1.2, $ut \notin E$.

Subcase 1.1. Gal(G) has a C_7 on db, ba, eb, bc, ct, tu, td or a C_7 on uc, cd, ds, sf, fe, eb, bc or a C_7 on st, te, at, td, db, bu, bs or a C_9 on bu, uf, fe, fc, cb, bs, st, te, tu or a C_7 on de, eu, us, sa, ac, cf, fe or a C_9 on eb, ba, db, bu, uf, au, ue, ed, ef.

Subcase 1.2. Gal(G) has a C_7 on ft, ta, et, tc, cb, cf, fu or a C_7 on uf, ft, td, db, ba, be, ef or a C_7 on bc, cu, ue, ed, ds, sf, sb or a C_5 on bu, uf, ft, td, db.



F_2

FIGURE 9

Case 2. There are vertices s, u with $s \neq c$, $sa \in E$, $sc \notin E$, $u \neq t$, $ub \in E$, and $ut \notin E$. Now Gal(G) has a C_7 on sa, ab, be, ef, ed, dc, ca or a C_7 on ca, as, ta, ab, be, bc, cf or a C_7 on ab, bd, sb, bc, ct, te, ta or a C_5 on se, ef, fc, ca, as or a C_9 on sa, ac, cd, de, ef, eb, bc, bs, sf or a C_7 on dt, ta, et, tc, cb, bu, bd or a C_7 on cd, du, dt, ta, te, ef, fc or a C_7 on bc, ct, uc, cd, de, ef, eb or a C_7 on ed, du, dt, ta, ab, be, ef or a C_7 on fe, eb, bs, st, tc, cu, ue or a C_7 on dc, ca, uc, ct, te, ef, ed or a C_9 on ua, at, ab, be, ef, fc, ct, cu, ue.

Case 3. We may assume that $tf \notin E$ (else Gal(G) has a C_5 on ta, ab, be, ef, ft). There exists a chordless path $t_1 t_2 \cdots t_k$ such that $t_1 = t$, $t_k = f$, and

$$t_i \neq c, \qquad t_i c \notin E \qquad \text{whenever} \quad 1 < i < k.$$

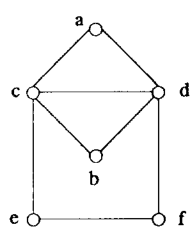
We shall distinguish between two subcases. In Subcase 3.1, $k = 3$; in Subcase 3.2, $k > 3$.

Subcase 3.1. Gal(G) has a C_7 on $t_2 t$, ta, ab, be, ef, fc, ct or a C_7 on $t_2 a$, ab, be, ef, ed, dc, ca or a C_9 on dc, cf, ft_2, $t_2 b$, be, ba, at, td, de or a C_7 on bt_2, $t_2 t$, et_2, $t_2 a$, ac, cf, cb.

Subcase 3.2. Gal(G) has a C_5 on ct, tt_2, $t_2 t_3$, $t_3 f$, fc or a C_7 on $t_2 t$, ta, ab, be, ef, fc, ct or a C_7 on $t_2 t$, tc, cb, be, ef, ed, dt or a C_5 on tt_2, $t_2 e$, ef, fc, ct or a C_7 on $t_3 t_2$, $t_2 t$, tc, cf, fe, ed, dt_2, or G has an F_1 on e, d, c, t, t_2, t_3, or Gal(G) has a C_7 on fe, et_3, $t_3 t_2$, $t_2 a$, ac, cd, de or a C_7 on $t_3 a$, ab, be, ef, ed, dc, ca or a C_7 on ca, at_3, ta, ab, be, bc, cf. ∎

CLAIM 3. *If G contains an induced F_3 (see Fig.* 10), *then* Gal(G) *contains an odd hole.*

Proof. There exists a chordless path $a_1 a_2 \cdots a_k$ such that $a_1 = a$, $a_k = b$, and k *is even.* If $ca_2 \notin E$ and $ca_{k-1} \notin E$, then Gal(G) has a C_{k+1} on ca, $aa_2, \ldots, a_{k-1} b$, bc; hence c is adjacent to at least one of a_2 and a_{k-1}. Now symmetry allows us to distinguish among seven cases:



F_3

FIGURE 10

1. $ca_2 \in E$, $da_2 \notin E$;
2. $k=4$ and $ca_2 \in E$, $ca_3 \in E$, $da_2 \in E$, $da_3 \in E$, $ea_3 \in E$, $fa_3 \in E$;
3. $k=4$ and $ca_2 \in E$, $ca_3 \in E$, $da_2 \in E$, $da_3 \in E$, $ea_3 \in E$, $fa_3 \notin E$;
4. $k=4$ and $ca_2 \in E$, $ca_3 \in E$, $da_2 \in E$, $da_3 \in E$, $ea_3 \notin E$, $fa_3 \notin E$;
5. $k=4$ and $ca_2 \in E$, $ca_3 \notin E$, $da_2 \in E$, $da_3 \notin E$;
6. $k \geqslant 6$ and $ca_2 \in E$, $ca_{k-1} \in E$, $da_2 \in E$, $da_{k-1} \in E$;
7. $k \geqslant 6$ and $ca_2 \in E$, $ca_{k-1} \notin E$, $da_2 \in E$, $da_{k-1} \notin E$.

Case 1. G has an F_1 on e, c, a_2, a, d, b, or $\mathrm{Gal}(G)$ has a C_5 on fe, ea_2, a_2a, ad, df, or G has an F_2 on b, d, c, a, f, a_2.

Case 2. We may assume that $ea_2 \notin E$ and $fa_2 \notin E$: otherwise $\mathrm{Gal}(G)$ has a C_9 on aa_2, a_2a_3, a_3b, ea_3, a_3d, da, bd, da_2, a_2e or a C_9 on aa_2, a_2a_3, a_3b, fa_3, a_3c, ca, bc, ca_2, a_2f. There is a vertex t with $t \neq d$, $tb \in E$, and $td \notin E$. We shall distinguish between two subcases. In Subcase 2.1, $te \in E$; in Subcase 2.2, $te \notin E$.

Subcase 2.1. $\mathrm{Gal}(G)$ has a C_9 on tb, bd, df, dc, ce, ac, ca_3, a_3f, a_3b or a C_{11} on et, tb, bd, da, da_3, a_3e, ba_3, a_3f, a_3c, ca, ce or a C_5 on te, ef, fd, db, bt or a C_{11} on da_3, a_3e, ba_3, a_3f, a_3c, ca, ct, tf, tb, bd, da or a C_{13} on a_3d, da, db, bt, tf, tc, cd, ec, ca, ca_3, a_3f, ba_3, a_3e.

Subcase 2.2. $\mathrm{Gal}(G)$ has a C_9 on tb, bd, df, dc, ce, ac, ca_3, a_3f, a_3b, or G has an F_1 on e, a_3, t, b, d, a_2, or $\mathrm{Gal}(G)$ has a C_7 on cb, bt, ta_2, a_2d, df, fe, ec or a C_9 on ca_3, a_3f, ta_3, a_3d, da, fd, dc, ce, ca or a C_{11} on ac, ct, tf, tb, bd, da, da_3, a_3e, ba_3, a_3f, a_3c or a C_{11} on ft, tc, ce, ac, ca_3, a_3f, ba_3, a_3e, a_3d, da, at.

Case 3. We may assume first that $fa_2 \in E$ (else G has an F_1 on f, d, a, a_2, a_3, b), and then that $ea_2 \notin E$ (else the transformation $a \leftrightarrow b$, $a_2 \leftrightarrow a_3$ brings us to Case 2). There is a vertex t with $t \neq d$, $ta \in E$, and $td \notin E$. Now $\mathrm{Gal}(G)$ has a C_7 on ta, aa_2, a_2a_3, a_3e, ef, fd, da or a C_7 on ta_2, a_2a_3, a_3e, ef, fd, db, da_2 or a C_7 on ta, ac, ce, ef, ea_3, a_3d, da or a C_7 on a_2c, cb, tc, cd, df, fe, fa_2 or a C_7 on at, tb, bd, df, fe, ea_3, a_3t or a C_5 on te, ef, fd, db, bt or a C_9 on ft, tc, cd, ce, ef, ea_3, a_3d, da, at.

Case 4. We may assume that $ea_2 \in E$ and $fa_2 \in E$: otherwise G has an F_1 on e, c, a, a_2, a_3, b or an F_1 on f, d, a, a_2, a_3, b. Now the transformation $a \leftrightarrow b$, $a_2 \leftrightarrow a_3$ brings us to Case 2.

Case 5. There is a vertex t with $t \neq d$, $ta \in E$, and $td \notin E$. We may assume first that $ea_2 \in E$ and $fa_2 \in E$: otherwise $\mathrm{Gal}(G)$ has a C_7 on a_2c, ce, ef, fd, db, ba_3, a_3a_2 or a C_7 on a_2d, df, fe, ec, cb, ba_3, a_3a_2. If $ea_3 \notin E$ and $fa_3 \notin E$, then G has an F_1 on a_3, a_2, a, c, e, f; hence symmetry allows us to assume that $ea_3 \in E$. Now Gal(G) has a C_5 on db, ba_3, a_3e, ef, fd or a C_7 on ta, ac, cb, ba_3, a_3f, fd, da or a C_7 on ta, ad, df, fa_3, a_3b, a_3a_2, a_2a or a C_7 on da_2, a_2t, fa_2, a_2c, cb, ba_3, bd or a C_7 on db, ba_3, a_3e, ec, ct, tf, fd or a C_9 on ft, ta, ad, db, ba_3, a_3e, ec, cd, ct.

Case 6. If $ea_2 \notin E$, $ea_{k-1} \notin E$, $fa_2 \notin E$, $fa_{k-1} \notin E$, then we may replace a by a_2 and b by a_{k-2}, and use induction on k; hence symmetry allows us to assume that $ea_{k-1} \in E$. Next, we may assume that $fa_{k-1} \in E$ (else G has an F_2 on a, c, d, a_{k-1}, e, f), and that $ea_2 \notin E$ (else G has an F_2 on b, c, d, a_2, e, f, or Gal(G) has a C_{11} on aa_2, a_2f, a_2c, ca_{k-1}, ac, ce, cd, df, bd, da_2, a_2e), and then that $fa_2 \notin E$ (else G has an F_2 on b, d, c, a_2, f, e). Now, symmetry allows us to distinguish between two subcases. In Subcase 6.1, $da_{k-2} \in E$; in Subcase 6.2, $ca_{k-2} \notin E$, $da_{k-2} \notin E$.

Subcase 6.1. We may assume that $da_{k-3} \notin E$, $ca_{k-2} \in E$, $ea_{k-2} \notin E$, $fa_{k-2} \notin E$: otherwise G has an F_1 on a, d, b, a_{k-1}, a_{k-2}, a_{k-3}, or Gal(G) has a C_7 on $a_{k-1}c$, ca_2, ec, cd, da_{k-2}, $a_{k-2}a_{k-3}$, $a_{k-2}a_{k-1}$ or a C_7 on $a_{k-2}a_{k-1}$, $a_{k-1}b$, ea_{k-1}, $a_{k-1}d$, da, da_{k-2}, $a_{k-2}a_{k-3}$, or G has an F_2 on a, d, c, a_{k-2}, f, e. Now we may replace b by a_{k-2} and use induction on k.

Subcase 6.2. We may assume that $ca_3 \in E$ and $da_3 \in E$ (else Gal(G) has a C_{k-1} on xa_2, $a_2a_3, \ldots, a_{k-2}a_{k-1}$, $a_{k-1}x$ with c or d in place of x). If $ea_3 \notin E$ and $fa_3 \notin E$, then we may replace a by a_3 and use induction on k; hence symmetry allows us to assume that $ea_3 \in E$. Now G has an F_2 on a, c, d, a_3, e, f, or Gal(G) has a C_{11} on fa_3, a_3c, ca_{k-1}, a_2c, ce, cd, df, bd, da_3, a_3e, a_2a_3.

Case 7. We may assume that $ca_3 \notin E$ (else Gal(G) has a C_{k-1} on ca_3, $a_3a_4, \ldots, a_{k-1}b$, bc, or G has an F_1 on b, c, a, a_2, a_3, a_4); the same argument with d in place of c allows us to assume $da_3 \notin E$. Now let i be the smallest subscript with $i \geqslant 3$ and $ca_i \in E$. We may assume that $i = k$ (else Gal(G) has a C_{i+3} on bd, da_2, $a_2a_3, \ldots, a_ic$, cb, ba_{k-1} as well as a C_i on ca_2, $a_2a_3, \ldots, a_{i-1}a_i$, a_ic); to put it differently, $ca_i \notin E$ whenever $3 \leqslant i \leqslant k-1$. The same argument with c and d switched allows us to assume that $da_i \notin E$ whenever $3 \leqslant i \leqslant k-1$. (In particular, all of $a_1, a_2, \ldots, a_{k-1}$ are distinct from e and f.) Next, we may assume that $ea_2 \in E$ (else Gal(G) has a C_{k+3} on ca_2, $a_2a_3, \ldots, a_{k-1}b$, bd, df, fe, ec); the same argument with the transformation $c \leftrightarrow d$, $e \leftrightarrow f$ allows us to assume that $fa_2 \in E$. There is a vertex t with $t \neq a_2$, $tf \in E$, and $ta_2 \notin E$. Now Gal(G) has a C_7 on da_2, a_2e, aa_2, a_2f, ft, fd, db or a C_{11} on td, da_2, a_2e, aa_2, a_2f, a_2c, cb, ce, ef, fd, da or a C_9 on td, da_2, a_2e, aa_2, a_2f, a_2c, cb, ec, cd or a C_9 on bc, ct, tf, fa_2, a_2a, ea_2, a_2d, db, ba_{k-1} or a C_7 on $a_{k-1}b$, bt, ta, aa_2, a_2e, a_2d, db or a C_7 on fa_2, a_2a, ea_2, a_2d, dt, ta_{k-1}, tf or a C_7 on fa_2, a_2a, ea_2, a_2d, db, ba_{k-1}, $a_{k-1}f$. ▮

CLAIM 4. *If G contains an induced F_4 (see Fig.* 11), *then* Gal(G) *contains an odd hole.*

Proof. There exists a chordless path $a_1 a_2 \cdots a_k$ such that $a_1 = a$, $a_k = e$, and

$$a_i \neq c \text{ and } a_i c \notin E \qquad \text{whenever} \quad 1 < i < k.$$

We may assume that $ba_2 \in E$, $ba_3 \in E$, and $k \leqslant 4$: otherwise Gal(G) has a C_7 on ac, cd, fc, cb, be, ba, aa_2 or a C_7 on ac, cd, fc, cb, ba_2, a_2a_3, a_2a, or G has an F_1 on e, b, a_3, a_2, a, c. There exists a chordless path $d_1 d_2 \cdots d_l$ such that $d_1 = d$, $d_l = f$, and

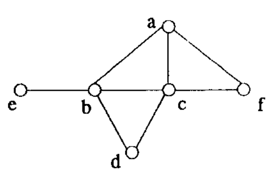
$$d_i \neq a \text{ and } d_i a \notin E \qquad \text{whenever} \quad 1 < i < l.$$

We shall distinguish among six cases:

1. $l = 3$ and $da_2 \in E$;
2. $l = 3$ and $da_2 \notin E$;
3. $l > 3$ and $cd_{l-1} \in E$;
4. $l > 3$ and $cd_{l-1} \notin E$, $cd_2 \in E$, $fa_2 \in E$;
5. $l > 3$ and $cd_{l-1} \notin E$, $cd_2 \in E$, $fa_2 \notin E$;
6. $l > 3$ and $cd_{l-1} \notin E$, $cd_2 \notin E$.

Case 1. Note that the eight vertices a, b, c, d, e, f, a_2, d_2 are all distinct. Gal(G) has a C_5 on ab, bd, dd_2, d_2f, fa, or G has an F_3 on e, b, d_2, d, f, a or an F_1 on e, b, a, c, d, d_2, or Gal(G) has a C_9 on fa, aa_2, ac, cd, fc, cb, be, bd_2, d_2f, or G has an F_2 on e, b, a_2, d, f, c or an F_2 on e, a_3, b, a_2, c, f or an F_2 on a_3, a_2, b, d, c, f, or Gal(G) has a C_9 on a_3d, dd_2, d_2f, fa, ab, be, bc, cf, cd or a C_9 on ea_3, a_3d_2, d_2f, d_2b, be, bc, cf, cd, da_3. (Note that $a_3 \neq d_2$ since $a_3c \notin E$, $cd_2 \in E$.)

Case 2. Note that the eight vertices a, b, c, d, e, f, a_2, d_2 are all distinct. Gal(G) has a C_9 on a_2b, bd, ba, af, a_2a, ac, cd, fc, cb or a C_5 on ab, bd, dd_2, d_2f, fa or a C_5 on aa_2, a_2a_3, a_3d, dc, ca, or G has an F_2 on d, b, c, a, f, a_3 or an F_3 on a_3, b, d_2, d, f, a or an F_2 on a_3, b, a_2, a, f, d_2, or Gal(G) has a C_9 on dd_2, d_2a_2, a_2a_3, fa_2, a_2b, bd, ba, af, fd_2.



F_4

FIGURE 11

Case 3. Note that the eight vertices a, b, c, d, e, f, a_2, d_{l-1} are all distinct. We may assume first that $ed_{l-1} \notin E$ (else Gal(G) has a C_5 on ab, be, ed_{l-1}, $d_{l-1}f$, fa or a C_9 on eb, bd, bd_{l-1}, $d_{l-1}f$, ed_{l-1}, $d_{l-1}c$, cd, fc, cb), and then that $cd_{l-2} \in E$ (else Gal(G) has a C_7 on eb, ba, af, fd_{l-1}, $d_{l-1}d_{l-2}$, $d_{l-1}c$, cd or a C_9 on dc, cd_{l-1}, $d_{l-1}d_{l-2}$, $d_{l-1}f$, fa, ab, be, bc, cf), and then that $l=4$ (else G has an F_1 on d, c, a, f, d_{l-1}, d_{l-2}), and so $cd_2 \in E$, $cd_3 \in E$, $ed_3 \notin E$, $fd_3 \in E$. We shall distinguish between two subcases. In Subcase 3.1, $bd_2 \in E$; in Subcase 3.2, $bd_2 \notin E$.

Subcase 3.1. We may assume that $ed_2 \in E$ (else we may replace d by d_2 and use induction on l). Now Gal(G) has a C_5 on ab, bd_2, d_2d_3, d_3f, fa or a C_9 on ed_2, d_2d_3, d_3f, fa, ab, be, bc, cf, cd_2.

Subcase 3.2. Gal(G) has a C_7 on ca, aa_2, fa, ab, be, bc, cd_2 or a C_7 on af, fd_3, a_2f, fc, cb, be, ba or a C_9 on dd_2, d_2d_3, d_3a_2, a_2a, ac, cd_2, cb, be, bd, or G has an F_2 on d, c, d_2, d_3, a_2, a, or Gal(G) has a C_7 on d_3a_2, a_2d, a_2f, a_2d_2, d_2c, ca, cd_3.

Case 4. Note that the vertices a, b, c, d, f, a_2, $d_2, ..., d_{l-1}$ are all distinct. We may assume that $da_2 \in E$ (else Gal(G) has a C_7 on cb, ba_2, bd, ba, af, fd_{l-1}, fc). Now let m be the largest subscript with $2 \leqslant m \leqslant l-2$ and $cd_m \in E$; since Gal(G) has a C_{l-m+2} on cd_m, $d_md_{m+1}, ..., d_{l-2}d_{l-1}$, $d_{l-1}f$, fc, we may assume that *$l-m$ is even*. Next, we may assume that $a_2d_{l-1} \in E$ (else Gal(G) has a C_{l-m+5} on da_2, a_2a, ac, cd_m, $d_md_{m+1}, ..., d_{l-2}d_{l-1}$, $d_{l-1}f$, fa_2), and then that $a_2d_{l-2} \in E$ (else Gal(G) has a C_{l-m+3} on ac, cd_m, $d_md_{m-1}, ..., d_{l-2}d_{l-1}$, $d_{l-1}a_2$, a_2a), and then that $l=4$ (else G has an F_1 on d, a_2, a, f, d_{l-1}, d_{l-2}). Now Gal(G) has a C_7 on bc, cd_2, d_2d_3, d_3f, fa, ab, be or a C_5 on ab, bd_2, d_2d_3, d_3f, fa, or G has an F_2 on d, b, c, a, f, d_3.

Case 5. Note that the vertices a, b, c, d, f, $a_2, d_2, ..., d_{l-1}$ are all distinct. We may assume first that $da_2 \in E$ (else Gal(G) has a C_7 on cb, ba_2, db, ba, af, fd_{l-1}, fc), and that $bd_2 \in E$ (else Gal(G) has a C_7 on ba, af, a_2a, ac, cd_2, cb, be; note that $e \neq d_2$ since $ce \notin E$, $cd_2 \in E$), and then that $a_2d_2 \in E$ (else Gal(G) has a C_7 on cb, ba_2, d_2b, ba, af, fd_{l-1}, fc). Now let m be the largest subscript with $2 \leqslant m \leqslant l-2$ and $cd_m \in E$; since Gal(G) has a C_{l-m+2} on cd_m, $d_md_{m+1}, ..., d_{l-2}d_{l-1}$, $d_{l-1}f$, fc, we may assume that *$l-m$ is even*. Next, we may assume that $m=2$ (else Gal(G) has a C_{l-m+5} on dc, cd_m, $d_md_{m+1}, ..., d_{l-2}d_{l-1}$, $d_{l-1}f$, fa, aa_2, a_2d), and so *l is even* and $cd_i \notin E$ whenever $3 \leqslant i \leqslant l-1$. Now Gal($G$) has a C_{l+1} on aa_2, a_2d_2, $d_2d_3, ...$, $d_{l-2}d_{l-1}$, $d_{l-1}f$, fa, or G has an F_2 on d_3, d_2, a_2, d, c, a.

Case 6. Since Gal(G) has a C_{l+1} on cd, $dd_2, ..., d_{l-2}d_{l-1}$, $d_{l-1}f$, fc, we may assume that *l is odd*. Next, we may assume that $bd_2 \in E$ (else Gal(G) has a C_7 on d_2d, db, ba, af, fd_{l-1}, fc, cd), and then that $bd_3 \notin E$ (else

Gal(G) has a C_l on ab, bd_3, $d_3d_4, \ldots, d_{l-2}d_{l-1}$, $d_{l-1}f$, fa, or G has an F_1 on a, b, d, d_2, d_3, d_5), and that $ed_2 \in E$ (else G has an F_1 on e, b, d_2, d, c, a; note that $e \neq d_2$ since $dd_2 \in E$, $de \notin E$). There is a vertex t with $t \neq b$, $td \in E$, and $tb \notin E$. Now Gal(G) has a C_7 on d_2b, bc, eb, bd, dt, dd_2, d_2d_3 or a C_7 on dd_2, d_2e, td_2, d_2b, bc, cf, cd or a C_7 on td, dc, cf, fd_{l-1}, fa, ab, bd or a C_7 on d_2t, tc, et, td, db, ba, bd_2. ∎

CLAIM 5. *If G contains an induced F_5 (see Fig. 12), then* Gal(G) *contains an odd hole.*

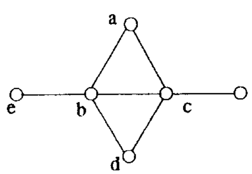
Proof. There exists a chordless path $a_1a_2 \cdots a_k$ such that $a_1 = a$, $a_k = d$, and k *is even.* If $ca_2 \notin E$ and $ca_{k-1} \notin E$, then Gal(G) has a C_{k+1} on ca, $aa_2, \ldots, a_{k-1}d$, dc; hence c is adjacent to at least one of a_2 and a_{k-1}. Now symmetry allows us to distinguish among five cases:

1. $ba_2 \notin E$, $ca_2 \in E$;
2. $k = 4$ and $ba_2 \in E$, $ba_3 \in E$, $ca_2 \in E$, $ca_3 \in E$;
3. $k = 4$ and $ba_2 \in E$, $ba_3 \notin E$, $ca_2 \in E$, $ca_3 \notin E$;
4. $k \geqslant 6$ and $ba_2 \in E$, $ba_{k-1} \in E$, $ca_2 \in E$, $ca_{k-1} \in E$;
5. $k \geqslant 6$ and $ba_2 \in E$, $ba_{k-1} \notin E$, $ca_2 \in E$, $ca_{k-1} \notin E$.

Case 1. There is a vertex t with $t \neq c$, $ta \in E$, and $tc \notin E$. Now G has an F_1 on f, c, d, b, a, a_2, or Gall(G) has a C_7 on ac, cd, a_2c, cb, eb, ba, at or a C_7 on ba, aa_2, ta, ac, cf, cb, be or a C_7 on aa_2, a_2f, ta_2, a_2c, cb, be, ab or a C_7 on a_2t, tb, ft, ta, ac, cd, ca_2.

Case 2. There is a vertex t with $t \neq b$, $ta_2 \in E$, and $tb \notin E$. If $fa_2 \notin E$ and $fa_3 \notin E$, then G has a F_1 on f, c, a, a_2, a_3, d; hence symmetry allows us to assume that $fa_3 \in E$. Now Gal(G) has a C_9 on db, ba_2, a_2f, aa_2, a_2a_3, a_3d, fa_3, a_3b, ba or a C_9 on fa_3, a_3b, ba, eb, bc, cf, ac, ca_3, a_3e, or G has an F_1 on e, b, a, a_2, a_3, d, or Gal(G) has a C_7 on a_3b, ba, db, ba_2, a_2t, a_2a_3, a_3f or a C_9 on a_2a_3, a_3d, ta_3, a_3b, be, bc, cf, ca_2, a_2e, or G has an F_3 or an F_2 on e, b, a_2, a, t, d, or Gal(G) has a C_9 on fa_3, a_3t, te, eb, bc, cf, ca_2, a_2e, a_2a_3, or G has an F_2 on f, t, a_3, d, b, e.

Case 3. Gal(G) has a C_7 on eb, ba_2, a_2a_3, a_3d, dc, cf, cd, or G has an F_3 or an F_2 on a, b, a_2, e, a_3, d.



F_5

FIGURE 12

Case 4. If $ea_2 \notin E$, $ea_{k-1} \notin E$, $fa_2 \notin E$, and $fa_{k-1} \notin E$, then we may replace a by a_2 and d by a_{k-1}, and use induction on k; hence symmetry allows us to assume that $fa_2 \in E$. Now G has an F_4 on a_2, b, c, d, e, f, or Gal(G) has a C_9 on fa_2, a_2b, bd, be, bc, cf, dc, ca_2, a_2e.

Case 5. We may assume that $ba_3 \notin E$ (else Gal(G) has a C_{k-1} on ba_3, $a_3a_4, ..., a_{k-2}a_{k-1}$, $a_{k-1}d$, db, or G has an F_1 on d, b, a, a_2, a_3, a_4); the same argument with c in place of b allows us to assume that $ca_3 \notin E$. Now let i be the smallest subscript with $i > 3$ and $ba_i \in E$. We may assume that $i = k$ (else Gal(G) has a C_{i+3} on $a_{k-1}d$, dc, ca_2, $a_2a_3, ..., a_{i-1}a_i$, a_ib, bd as well as a C_i on ba_2, $a_2a_3, ..., a_{i-1}a_i$, a_ib); to put it differently, $ba_i \notin E$ whenever $3 \leqslant i \leqslant k-1$. The same argument with b and c switched allows us to assume that $ca_i \notin E$ whenever $3 \leqslant i \leqslant k-1$. (In particular, all of a_2, $a_3, ..., a_{k-1}$ are distinct from e and f.) Next, we may assume that $ea_2 \in E$ (else Gal(G) has a C_{k+3} on eb, ba_2, $a_2a_3, ..., a_{k-2}a_{k-1}$, $a_{k-1}d$, dc, cf, cb); the same argument with the transformation $b \leftrightarrow c$, $e \leftrightarrow f$ allows us to assume that $fa_2 \in E$. Now Gal(G) has a C_7 on ca_2, a_2e, fa_2, a_2b, bd, da_{k-1}, dc. ∎

Claim 6. *If G contains an induced dart* (*see Fig.* 13), *then* Gal(G) *contains an odd hole.*

Proof. There is a vertex t with $t \neq b$, $tc \in E$, and $tb \notin E$. We may assume that $ta \in E$ and $td \in E$: otherwise G has an F_5 or an F_3 or an F_4 or an F_2 on a, b, c, d, e, t. There exists a chordless path $a_1a_2 \cdots a_k$ such that $a_1 = a$, $a_k = d$, and k *is even*. If $ca_2 \notin E$ and $ca_{k-1} \notin E$, then Gal(G) has a C_{k+1} on ca, $aa_2, ..., a_{k-1}d$, dc; hence c is adjacent to at least one of a_2 and a_{k-1}. Now symmetry allows us to distinguish among five cases:

1. $ba_{k-1} \notin E$;
2. $k = 4$ and $ba_2 \in E$, $ba_3 \in E$, $ca_2 \in E$, $ca_3 \in E$;
3. $k = 4$ and $ba_2 \in E$, $ba_3 \in E$, $ca_2 \in E$, $ca_3 \notin E$;
4. $k \geqslant 6$ and $ba_2 \in E$, $ba_{k-1} \in E$, $ca_2 \in E$, $ca_{k-1} \in E$;
5. $k \geqslant 6$ and $ba_2 \in E$, $ba_{k-1} \in E$, $ca_2 \in E$, $ca_{k-1} \notin E$.

Case 1. We may assume first that $ba_2 \in E$ (else Gal(G) has a C_{k+1} on ba, $aa_2, ..., a_{k-2}a_{k-1}$, $a_{k-1}d$, db), and that $ca_{k-1} \notin E$ (ekse G has an F_4 or

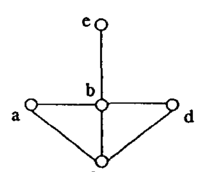


Figure 13

an F_2 on b, c, d, e, a, a_{k-1}; note that $e \neq a_{k-1}$ since $be \in E$, $ba_{k-1} \notin E$), and then that $ca_2 \in E$ (else Gal(G) has a C_{k+1} on ca, $aa_2, ..., a_{k-2}a_{k-1}$, $a_{k-1}d$, dc)), and that $ba_3 \notin E$ (else Gal(G) has a C_{k-1} on $ba_3, a_3a_4, ..., a_{k-2}a_{k-1}, a_{k-1}d, db$, or G has an F_1 on d, b, a, a_2, a_3, a_4; note that $k \geqslant 6$ when $ba_3 \in E$, $ba_{k-1} \notin E$); the same argument with c in place of b allows us to assume that $ca_3 \notin E$. Now Gal(G) has a C_7 on ct, ta_{k-1}, at, td, db, be, bc or a C_9 on eb, ba_2, a_2a_3, a_2c, cd, da_{k-1}, dt, ta, ab (note that $e \neq a_3$ since $be \in E$, $ba_3 \notin E$), or G has an F_3 or an F_2 on a, b, a_2, e, a_3, d or an F_4 on c, a_2, b, e, a_3, d or an F_4 on e, b, a_2, a, d, a_3.

Case 2. G has an F_1 on e, b, a, a_2, a_3, d, or Gal(G) a C_9 on dc, ca_2, a_2e, aa_2, a_2a_3, a_3d, ea_3, a_3c, ca, or G has an F_4 on e, c, a, a_2, a_3, d.

Case 3. There is a vertex s with $s \neq b$, $sa_3 \in E$, and $sb \notin E$. Now G has an F_1 on e, b, a, c, d, a_3, or Gal(G) has a C_7 on ct, ta_3, at, td, db, be, bc or a C_5 on at, td, da_3, a_3a_2, a_2a or a C_7 on sa_3, a_3b, ba, at, td, ta_2, a_2a_3 or a C_7 on db, be, cb, ba_3, a_3s, a_3d, dt or a C_7 on db, be, cb, ba_3, a_3s, sa, sd or a C_5 on at, ts, sa_3, a_3b, ba or a C_7 on bd, dt, a_3d, dc, ca_2, a_2s, a_2b or a C_9 on ac, cs, sa_3, a_3b, ba, bd dt, a_3d, dc.

Case 4. If $ba_3 \notin E$ and $ba_{k-2} \notin E$, then Gal(G) has an odd hole on ba_2, $a_2a_3, ...,$ $a_{k-2}a_{k-1}$, $a_{k-1}b$; hence symmetry allows us to assume that $ba_3 \in E$, and so $ba_4 \notin E$ (else G has an F_1 on d, b, a, a_2, a_3, a_4), and then that $ba_{k-2} \notin E$ (else Gal(G) has a C_{k-3} on $ba_3, a_3a_4, ..., a_{k-3}a_{k-2}, a_{k-2}b$, or G has an F_1 on a, b, d, a_{k-1}, a_{k-2}, a_{k-3}; note that $k \geqslant 8$ in the former case, when $ba_{k-3} \notin E$, $ba_3 \in E$), and then that $ca_3 \in E$ (else Gal(G) has a C_{k-1} on $ca_2, a_2a_3, ..., a_{k-2}a_{k-1}, a_{k-1}c$, or G has an F_4 or an F_2 on b, a_2, a_3, c, d, a_{k-2}), and then $ca_4 \notin E$ (else G has an F_1 on d, c, a, a_2, a_3, a_4), and that $ea_3 \in E$ (else we may use induction on k with a_3 in place of a; note that $e \neq a_3$ since $ca_3 \in E$, $ce \notin E$). Now G has an F_1 on d, b, a, a_2, a_3, e, or Gal(G) has a C_7 on ca_2, a_2e, aa_2, a_2a_3, a_3a_4, a_3c, cd.

Case 5. We may assume first that $ea_{k-1} \in E$ (else G has an F_1 on e, b, a, c, d, a_{k-1}; note that $e \neq a_{k-1}$ since $da_{k-1} \in E$, $de \notin E$), and that $ca_3 \notin E$ (else Gal(G) has a C_{k-1} on $ca_3, a_3a_4, ..., a_{k-1}d, dc$, or G has an F_1 on d, c, a, a_2, a_3, a_4). Now G has an F_1 on a_3, b, a, c, d, a_{k-1}, or Gal(G) has a C_7 on bd, dt, $a_{k-1}d$, dc, ca_2, a_2a_3, a_2b or a C_7 on db, be, cb, ba_{k-1}, $a_{k-1}t$, ta, td. ∎

4. Diamond-Free Raspail Graphs are Gallai-Perfect

Theorem 4. *Every diamond-free Raspail graph is Gallai-perfect.*

Proof. It will be convenient to prove the statement in the following form:

If G is diamond-free but not Gallai-perfect, then G is not Raspail. (6)

For this purpose, consider a diamond-free graph G that is not Gallai-perfect. Let S denote the set of vertices of an odd hole in Gall(G) with $|S|$ as small as possible; let E denote the set of edges of G; set $N_s(x) = \{y \mid xy \in S\}$. We only need prove that

$$N_S(x) \text{ is a stable set in } G \text{ for all vertices } x \text{ of } G. \tag{7}$$

If (7) is true, then each vertex of G is incident with at most two edges from S; hence the edges of G contained in S form an odd cycle with no triangular chord; thus (6) follows.

To verify (7), assume the contrary: there are three vertices x, a, b of G such that $ax \in S$, $bx \in S$, and $ab \in E$. If there is a vertex y of G such that $xy \in S$, $y \neq a$, $y \neq b$, then y is adjacent in G to either both or none of a and b (for otherwise G would contain an induced diamond on a, b, x, y).

Case 1. *$ay \in E$ and $by \in E$ whenever $xy \in S$, $y \neq a$, $y \neq b$.*

Since ax is adjacent to two vertices of Gal(G) in S, there are two vertices c and d of G such that $c \neq x$, $d \neq x$ and

$$ac \in S,\ ad \in S,\ cd \in E, \qquad cx \notin E, \qquad dx \notin E.$$

Next, we have

$$bc \notin E, \qquad bd \notin E:$$

for then G would contain an induced diamond on a, b, x, v with c or d in place of v. Hence we have

$$ab \notin S:$$

for then Gall(G) would contain a chordless C_4 on xa, ac, ba, ad (which are all in S).

By the same argument with a and b switched, there are vertices e and f of G such that c, d, e, f, x are distinct vertices and

$$be \in S,\ bf \in S,\ ef \in E,\ ae \notin E,\ af \notin E,\ ex \notin E, fx \notin E.$$

Enumerate the elements of S in the natural cyclic order as $e_1, e_2, ..., e_k, e_{k+1}, e_{k+2}, ..., e_{2l+1}$ with $3 < k < 2l-2$, such that $e_1 = ax$, $e_2 = ac$, $e_k = be$, $e_{k+1} = bx$, $e_{k+2} = bf$, $e_{2l+1} = ad$. Clearly, one of ab, e_2, $e_3, ..., e_k$ and ab, $e_{k+2}, e_{k+3}, ..., e_{2l+1}$ is an odd cycle in Gal(G); symmetry allows us to assume that ab, $e_2, e_3, ..., e_k$ is an odd cycle in Gal(G). By minimality of $|S|$, this cycle must have a chord. Since $e_2 e_3 \cdots e_k$ is a chordless path in Gal(G), one endpoint of this chord must be ab; symmetry allows us to assume that the other endpoint is az for some z in G. Trivially, $z \neq c$ and $z \neq d$. Now $xz \in E$ (for otherwise ax would be adjacent to three vertices of Gal(G) in S), and so $bz \in E$ (for otherwise G would contain an induced diamond on a, b, x, z). Hence ab is not adjacent to az in Gal(G), a contradiction.

Case 2. There is a vertex y of G such that $xy \in S$, $y \neq a$, $y \neq b$, and $ay \notin E$, $by \notin E$.

Since vertex ax of Gal(G) is adjacent to precisely two vertices of Gal(G) in S, it is adjacent either to some xz in S with $z \neq y$ or to some ac in S. In the first case, we must have $yz \in E$ (for otherwise xy would be adjacent to three vertices of Gal(G) in S); but then either G contains an induced diamond on b, x, y, z or Gal(G) contains a chordless C_4 on ax, xy, bx, xz (which are all in S), a contradiction. Hence

ax is adjacent in Gal(G) to some ac in S.

The same argument with a and b switched shows that

bx is *adjacent* in Gall(G) to some bd in S.

Since G is diamond-free, we have

$$bc \notin E \quad \text{and} \quad ad \notin E.$$

It follows that

$$ab \notin S:$$

for then Gal(G) would contain a chordless C_6 on yx, xa, ac, ba, db, bx (which are all in S).

Enumerate the elements of S in the natural cyclic order as $e_1, e_2, e_3, ..., e_{2l}, e_{2l+1}$ such that $e_1 = xy$, $e_2 = ax$, $e_3 = ac$, $e_{2l} = bd$, $e_{2l+1} = bx$. Clearly, $ab, e_3, e_4, ..., e_{2l}$ is an odd cycle in Gal(G). By minimality of $|S|$, this cycle must have a chord. Since $e_3 e_4 \cdots e_{2l}$ is a chordless path in Gal(G), one endpoint of this chord must be ab; symmetry allows us to assume that the other endpoint is az for some z in G. Trivially, $z \neq c$ and $z \neq d$. Now $xz \in E$ (for otherwise ax would be adjacent to three vertices of Gal(G) in S), and so $bz \in E$ (for otherwise G would contain an induced diamond on a, b, x, z). Hence ab is not adjacent to az in Gall(G), a contradiction. ∎

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